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Self-Synchronization of Nonlinear Oscillations in the Presence of Fluctuations

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A statistical mechanical theory is presented for the self-organization of a macroscopic oscillation with the presence of external fluctuations in a system of Van der Pol oscillators coupled through dissipative interactions. Starting from Langevin equations for the Van der Pol oscillators, the static and dynamic characteristics are studied. The threshold condition is given by the relative size between the fluctuation and the interaction. The transitions between synchronous and asynchronous phases are well discussed by a Landau-type equation. The steady state value of the order parameter and the onset time are compared between the theory and the computer experiments and a good agreement is obtained.

KEY WORDS: Self-synchronization; Van der Pol oscillator; fluctuations; phase transition; rhythm.

1. INTRODUCTION

The organization of the temporal order, the temporal organization, is one of the fundamental characteristics of a system far from thermal equilibrium.[1,2] In addition, the temporal order is more or less involved in a number of life phenomena.[2–5] Statistical mechanical studies for understanding the essential physics of the temporal organization will be important not only for statistical physics but also for various fields of biology. For the temporal organization in a statistical system, firstly, elementary oscillations must be present at the microscopic level, and secondly, such elementary oscillations must reach a temporal coherency. The resultant oscillation in a nonconservative system is known to be of the limit cycle

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type. The basic mechanism for producing the temporal coherency is what is called the entrainment or synchronization which is known as one of the characteristics of nonlinear oscillations.

Temporal organization can be classified into two according to the mechanism of the entrainment: self-synchronization and forced synchronization. In the former the synchronization spontaneously appears in a macroscopic system as a result of mutual interactions among elementary oscillators while the latter is derived from an external oscillation which may be linear or nonlinear. The self-synchronization will be very important in the formation of macroscopic rhythms in nature and is a good subject for statistical physics. Beating cardiac cell clusters,[6] glycolitic oscillations in yeast cell suspensions or in mitochondria,[4,7,8] and mitosis in Physarum[7] are well-known examples of self-synchronization where the elementary oscillators are actually identifiable. As examples of the self-synchronization in a continuous macroscopic system, we have the chemical oscillation in the Belousov–Zhabotinsky reaction,[9] the rhythmic contraction in Physarum plasmodium,[10] and so on. In regard to the forced synchronization, the synchronization of circadian rhythms to a periodic change of light intensity is an important example in the field of chronological biology.[5] Another interesting example can be seen in the forced synchronization in chemical oscillation, which has been studied extensively by Tomita's group from a theoretical point of view[11]: various aspects of oscillations, not only synchronizations but also chaotic behaviors.

Self-synchronization was studied theoretically by Wiener[12] in terms of the Wiener kernel, and the entrained spectrum of brain waves was derived by giving some specific characters to lower-order kernels. However, it is generally hard to relate Wiener kernels to statistical properties of the system to be derived from fundamental equations, for instance, equations of motion for elementary oscillators. In the place of Wiener kernels, Winfree introduced more tractable quantities such as a sensitivity function which was empirically introduced to give the response of the system, a shift in the frequencies of the oscillators, to a periodic stimulus.[13] He succeeded in showing the presence of a "threshold condition" for the self-synchronization of weakly interacting nonlinear oscillators with respect to the distribution of their frequencies. Phase-transition-like phenomena in self-synchronization were studied by Kuramoto, starting from kinetic equations derived for a reaction-diffusion system.[14] The threshold condition was given in terms of the relative size between the width of the frequency distribution and the magnitude of mutual interactions among the elementary oscillators. Details of the mutually entrained state and its process were clarified by Aizawa by means of computer simulations for interacting Van der Pol oscillators.[15] He also gave an explanation for properties of synchronization by utilizing a mean field approximation.

Elementary oscillations, which are generally of microscopic level in the system, are influenced, more or less, by the fluctuation. Hence, in addition to the entraining force, the temporal organization is governed by the fluctuation force. What is to be studied extensively in statistical physics on the temporal organization will be the influence of these two kinds of forces. In the case of forced synchronization, theoretical studies are comparatively advanced,[16] in particular, in the field of radio communications. The synchronization of a single nonlinear oscillator to an external oscillation in the presence of a fluctuating force was studied by Stratonovich.[17] In the case of self-synchronization in a statistical system, theoretical treatment will become more complex than in the case of forced synchronization since an N-body treatment is essentially required because of the cooperative behavior of elementary oscillators. General theory has not been published yet for the self-synchronization with random forces so far as the authors know, and little is known about the general properties in spite of their significance in various fields.

In the present paper, starting from coupled Langevin equations, we will treat the self-synchronization under the influence of external fluctuations in a system of Van der Pol oscillators interacting through dissipative coupling. In the next section, Langevin equations for Van der Pol oscillators as basic equations are coarse grained by means of the multi-time-scale method of Kawasaki[18] applied for the Fokker–Planck equation. The resultant equations, similar to rotating Van der Pol equations, are rewritten in terms of microscopic and macroscopic quantities in Section 3, which are treated by the consistent equation method by Kometani and Shimizu for two hierarchical systems.[19] In Section 4, a threshold is derived for the self-synchronization with respect to the relative size between the fluctuation and the interaction. In Section 5, a Landau-type equation is derived for the evolution of the order parameter, and the threshold condition for the self-synchronization derived from this equation coincides with that derived in Section 4. The process of the temporal organization of a macroscopic oscillation is studied in Section 5 by means of the dynamical scaling theory of Suzuki.[20] Section 6 is devoted to a comparison of the results of our theory and those of computer experiments. Good agreement is obtained. The conclusions made from our study are discussed in the last section in connection with other theories.

2. COARSE GRAINING OF BASIC EQUATIONS

The system we will study here is a N-body one composed of identical Van der Pol oscillators each of which is under a Gaussian white noise and interacts with the others through a dissipative interaction. The equation of

motion for the ith oscillator is given by

$$\ddot{x}_i + \omega_0^2 x_i - \epsilon\omega_0\kappa(1 - \beta x_i^2)\dot{x}_i - \frac{\epsilon\omega_0 R}{N}\sum_j(\dot{x}_j - \dot{x}_i) = f_i(t) \tag{1}$$

where the angular frequency ω_0 is common to the all oscillators, $\epsilon\omega_0\kappa$ with the smallness parameter κ gives the nonlinearity to the oscillation, and β is linked to the amplitude a_0 of the oscillation in the absence of the interaction and the random force by the relation $\beta = 4/a_0^2$. The dissipative interaction represented by the last term on the left-hand side (lhs) of Eq. (1) is assumed to be caused by some friction of which the strength is given by $\epsilon\omega_0 R/N$ with $R \geqslant 0$. This is equivalent to a mean-field-like interaction with the magnitude $\epsilon\omega_0 R$. The random force $f_i(t)$ on the right-hand side (rhs) is of external origin and will be assumed to be a Gaussian white as follows:

$$\langle f_i(t)\rangle = 0, \qquad \langle f_i(t) f_j(t')\rangle = 2\epsilon\omega_0^4 D\delta_{ij}\delta(t - t') \tag{2}$$

where $\langle \cdots \rangle$ denotes the ensemble average with respect to the random variable $f_i(t)$ and $\epsilon\omega_0^4 D$ is the corresponding diffusion coefficient, respectively.

Equation (1) can be rewritten as two first-order differential equations:

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= -\omega_0^2 x_i + \epsilon\omega_0\kappa(1 - \beta x_i^2)v_i + \frac{\epsilon\omega_0 R}{N}\sum_j(v_j - v_i) + f_i(t) \end{aligned} \tag{3}$$

which give $2N$ Langevin equations for the variables $\{x_i, v_i\}$. The Fokker–Planck equation for the probability density function $P(\{x_i\}, \{v_i\}, t)$ can be obtained from Eq. (3) as follows:

$$\frac{\partial}{\partial t}P(\{x_i\}, \{v_i\}, t) = HP(\{x_i\}, \{v_i\}, t) \tag{4a}$$

where the Fokker–Planck operator H can be separated into two parts according to the order of ϵ:

$$H = H_0 + \epsilon H_1 \tag{4b}$$

with

$$H_0 = \sum_i\left(\frac{\partial}{\partial x_i}v_i + \omega_0^2\frac{\partial}{\partial v_i}x_i\right) \tag{4c}$$

and

$$H_1 = \sum_i\left\{-\frac{\partial}{\partial v_i}\left[\omega_0\kappa(1 - \beta x_i^2)v_i + \frac{\omega_0 R}{N}\sum_j(v_j - v_i)\right] + \omega_0^4 D\frac{\partial^2}{\partial v_i^2}\right\} \tag{4d}$$

In Eqs. (4a)–(4c), H_0 gives the rapidly varying component which evolves with the time scale ω_0^{-1}, that corresponds to a harmonic oscillation, and H_1 gives the slowly varying component with the time scale $(\epsilon\omega_0)^{-1}$ caused by the nonlinearity, the interaction, and the external fluctuation.

A dynamical system may be conveniently broken down into a series of hierarchical subsystems from the rate of the time evolution: the time scales characterizing two successive hierarchical subsystems are different from each other by a factor, ϵ. For instance, the dynamics of the fastest rate, the harmonic oscillation, is described in terms of the time scale τ_0. The rate of the evolution of the next time scale τ_1 is smaller by a factor ϵ than that of the fastest change. The synchronization process will be this kind of dynamics, as is clear from Eqs. (4a)–(4d). Because of such breaking down, the time t governing the evolution of the whole system may be regarded as a function of independent variables $\{\tau_i\}$, or

$$\frac{d}{dt} = \sum_{i=0}^{\infty} \left(\frac{d\tau_i}{dt} \right) \frac{\partial}{\partial \tau_i} = \sum_{i=0}^{\infty} \epsilon^i \frac{\partial}{\partial \tau_i} \tag{5a}$$

Now, let us expand the probability distribution function $P(t)$ into a series in terms of ϵ:

$$P(t) = P_0(\{\tau_i\}) + \epsilon P_1(\{\tau_i\}) + \epsilon^2 P_2(\{\tau_i\}) + \cdots \tag{5b}$$

It should be noted that this expansion does not correspond to the breaking down of the time evolution of the system into hierarchical subsystems, but each term $P_j(\{\tau_i\})$ includes information on the dynamics of different hierarchical classes. The smallness parameter ϵ plays the role of the "weight" of information contained in each term $P_j(\{\tau_i\})$ within $P(t)$. Thus, when ϵ is very small, most of the information on the time evolution of the system will be condensed in the first term $P_0(\{\tau_i\})$ on the rhs.

Substituting Eqs. (5a) and (5b) into Eq. (4a), and then balancing both sides of the resulting equation with respect to ϵ, we obtain

$$\frac{\partial}{\partial \tau_0} P_0 = H_0 P_0 \tag{6a}$$

and

$$\left(\frac{\partial}{\partial \tau_0} - H_0 \right) P_1 = -\left(\frac{\partial}{\partial \tau_1} - H_1 \right) P_0 \tag{6b}$$

Equation (6a) denotes the time evolution of N "harmonic" oscillators, when the mutual interaction and the fluctuations are absent. So the equation describes deterministic motions. The operator H_1 representing the effects of the nonlinearity, the interaction, and the fluctuation is found in Eq. (6b), which means that information on the self-synchronization under external

fluctuations exists in the probability density function P_0 and P_1. Equations (6a) and (6b) can be solved formally as

$$P_0(\tau_0 - s, \tau_1) = e^{-sH_0} P_0(\tau_0, \tau_1) \tag{7a}$$

and

$$P_1(\tau_0, \tau_1) = \int_0^{\tau_0} ds\, e^{sH_0} \left[H_1 P_0(\tau_0 - s, \tau_1) - \frac{\partial}{\partial \tau_1} P_0(\tau_0 - s, \tau_1) \right] \tag{7b}$$

Substituting Eq. (7a) into (7b), we obtain

$$P_1(\tau_0, \tau_1) = \tau_0 \left(H_1^{\tau_0} - \frac{\partial}{\partial \tau_1} \right) P_0(\tau_0, \tau_1) \tag{8a}$$

with

$$H_1^{\tau_0} \equiv \tau_0^{-1} \int_0^{\tau_0} ds\, e^{sH_0} H_1 e^{-sH_0} \tag{8b}$$

In Eq. (8a) we take a limit $\tau_0 \to \infty$, keeping the time scale τ_1 unchanged. The physical meaning of this procedure is as follows. Firstly, it leads to $\epsilon = \tau_1/\tau_0 \to 0$, which shows that information concerning the self-synchronization process is exclusively condensed in P_0. Secondly, the harmonic oscillations in the time scale τ_0 are excluded and we can pay attention only to the dynamics in the time scale τ_1. Assuming that $P_0(\tau_0, \tau_1)$ converges to a finite value, we obtain from Eqs. (8a) and (8b)

$$\frac{\partial}{\partial \tau_1} \bar{P}_0(\tau_1) = \bar{H}_1 \bar{P}_0(\tau_1) \tag{9a}$$

with

$$\bar{H}_1 = \lim_{\tau_0 \to \infty} H_1^{\tau_0} \tag{9b}$$

The operator $\bar{H}_1$ may be interpreted as a coarse-grained stochastic operator of H_1 in regard to the rapidly varying motion, and it gives the probability density function $\bar{P}_0$ for the slowly varying motion.

To obtain the exact expression of $\bar{H}_1$, one must first calculate the term $e^{sH_0} H_1 e^{-sH_0}$, which can be expressed in terms of products of quantities, $q_0(-s) \equiv e^{sH_0} q e^{-sH_0}$. On the other hand, $q_0(-s)$ is obtained by solving the equation of motion for a variable, q,

$$\dot{q} = [q, H_0] \tag{10}$$

where $[\,,\,]$ denotes the commutator. This equation gives the evolution of the quantity q in the absence of the nonlinearity and the fluctuation. For instance, for $q = x_i$ and v_i,

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= -\omega_0^2 x_i \end{aligned} \tag{11}$$

from which one obtains

$$x_{i0}(-s) = a_i\cos(-\omega_0 s + \phi_i)$$
$$= x_i\cos\omega_0 s - \frac{v_i}{\omega_0}\sin\omega_0 s \tag{12a}$$
$$v_{i0}(-s) = -a_i\omega_0\sin(-\omega_0 s + \phi_i)$$
$$= x_i\omega_0\sin\omega_0 s + v_i\cos\omega_0 s \tag{12b}$$

with

$$a_i = \left[x_i^2 + (v_i/\omega_0)^2\right]^{1/2} \tag{13a}$$
$$\phi_i = -\tan^{-1}(v_i/\omega_0 x_i) \tag{13b}$$

Clearly, $\{x_{i0}\}, \{v_{i0}\}$ represent harmonic oscillations. Similarly, one finds

$$e^{sH_0}\frac{\partial}{\partial v_i}e^{-sH_0} = \frac{\sin\omega_0 s}{\omega_0}\cdot\frac{\partial}{\partial x_i} + \cos\omega_0 s\cdot\frac{\partial}{\partial v_i} \tag{14}$$

By substituting Eqs. (12a), (12b), and (14) into Eq. (8b) and by taking the limit $\tau_0\to\infty$, the explicit form of $\bar{H}_1$ is found as follows:

$$\bar{H}_1 = \left\{\frac{\omega_0^2 D}{2}\left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2}\right) - \frac{\omega_0\kappa}{2}\left(\frac{\partial}{\partial x_i}x_i + \frac{\partial}{\partial y_i}y_i\right)\left[1 - \frac{\beta}{4}(x_i^2 + y_i^2)\right]\right.$$
$$\left. - \frac{\omega_0 R}{2N}\sum_j\left[\frac{\partial}{\partial y_i}(y_j - y_i) + \frac{\partial}{\partial x_i}(x_j - x_i)\right]\right\} \tag{15}$$

where $y_i = v_i/\omega_0$. Langevin equations for the slowly varying motions of $\{x_i\}$ and $\{y_i\}$ can be derived from this Fokker–Planck operator:

$$\dot{x}_i = \frac{\kappa}{2}x_i\left[1 - \frac{\beta}{4}(x_i^2 + y_i^2)\right] + \frac{R}{2N}\sum_j(x_j - x_i) + g_{xi}(t) \tag{16a}$$

$$\dot{y}_i = \frac{\kappa}{2}y_i\left[1 - \frac{\beta}{4}(x_i^2 + y_i^2)\right] + \frac{R}{2N}\sum_j(y_j - y_i) + g_{yi}(t) \tag{16b}$$

where the random force $g_{zi}(t)$ with $z = x, y$ is characterized as

$$\langle g_{zi}(t)\rangle = 0, \qquad \langle g_{zi}(t)\, g_{z'j}(t')\rangle = D\delta_{zz'}\delta_{ij}\delta(t - t') \tag{17}$$

Equations (16a) and (16b) describe motions of the oscillators where the component of harmonic oscillations is excluded from the basic equations. In other words, it gives the motions of "rotating Van der Pol oscillators with the interaction and the fluctuation. Therefore, it can be conveniently considered that Eqs. (16a) and (16b) represent the motion of Eq. (1) observed on a coordinate rotating with the angular frequency ω_0. It should be noted that, in Eqs. (16) and (17) and hereafter, the time scale is denoted by $\omega_0\tau_1$, while in Eq. (15) it is denoted by τ_1.

3. SEPARATION OF MICRO- AND MACROSCOPIC QUANTITIES

In this section we will derive a set of equations describing the process of self-synchronization. The equations are one for the time evolution of the order parameter and ones for the change of internal fluctuations around the order parameter. These hierarchical subsystems, the order parameter and internal fluctuations, are linked to each other by means of a feedback loop of interactions, which can be described by a set of consistent equations as shown by Kometani and Shimizu.[19]

The stochastic variables $x_i(t)$ and $y_i(t)$ can be expressed as

$$\begin{aligned} x_i(t) &= X(t) + \xi_i(t) \\ y_i(t) &= Y(t) + \eta_i(t) \end{aligned} \tag{18}$$

where $X(t)$ and $Y(t)$ are macroscopic variables, which may be taken for a large N as follows:

$$X(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t) \qquad \text{and} \qquad Y(t) = \frac{1}{N}\sum_{i=1}^{N} y_i(t) \tag{19}$$

and $\xi_i(t)$ and $\eta_i(t)$ are microscopic variables, fluctuating deviations from the "macroscopic oscillations" represented in terms of $X(t)$ and $Y(t)$.

We may assume that at the initial time $t = 0$, N oscillators are uniformly distributed in their phases. Then, it will be reasonable to assume that during the time course of the synchronization the statistical distribution of the microscopic oscillations is symmetric with respect to the phase Φ_0 in the following sense. That is, the oscillators give rise to synchronized oscillation with the phase angle Φ_0, keeping the phase distribution symmetric around Φ_0. The assumption of this symmetric distribution in the phase may be accepted without loss of generality. Hence,

$$Y(t) = -\frac{1}{N}\sum a_i(t)\sin\Delta\phi_i(t) \simeq 0 \tag{20a}$$

and

$$X(t) = \frac{1}{N}\sum a_i(t)\cos\Delta\phi_i(t) \tag{20b}$$

where $\Delta\phi_i(t) \equiv \phi_i(t) - \Phi_0$. Though the macroscopic oscillation is denoted in terms of the variables $X(t)$ and $Y(t)$, the use of the condition (20a) for the symmetric phase distribution can reduce the number of the variables, leaving only $X(t)$ as a macroscopic variable. The assumption of the uniform phase distribution of the oscillators at $t = 0$ may be written as $X(0) = 0$. In the completely synchronized state, we have $\phi_i = \Phi_0$, leading to $X(t) = (1/N)\sum a_i(t) \equiv A(t)$ and $X(t)$ has the maximum value. These situations allow us to use $X(t)$ as the order parameter representing the degree of self-synchronization.

From Eqs. (18) and (20), we have

$$x_i(t) = X(t) + \xi_i(t) \tag{21a}$$

$$y_i(t) = \eta_i(t) \tag{21b}$$

Substituting Eqs. (21a) and (21b) into Eq. (16a) and then taking the average of the resulting equations with regard to i, we obtain the evolution equation for the order parameter $X(t)$:

$$\dot{X} = \frac{\kappa}{2}\left[1 - \frac{\beta}{4}(3\sigma_x + \sigma_y)\right]X - \frac{\beta\kappa}{8}X^3 + G(t) \tag{22}$$

where

$$\sigma_x(t) \equiv \sum_i \xi_i^2(t)/N, \qquad \sigma_y(t) \equiv \sum_i \eta_i^2(t)/N \tag{23a}$$

and

$$\langle G(t)\rangle = 0, \qquad \langle G(t)G(t')\rangle = \frac{D}{N}\delta(t - t') \tag{23b}$$

Subtracting Eq. (22) from Eq. (16a), we obtain

$$\begin{aligned}\dot{\xi} &= \frac{1}{2}\left(\kappa - R - \frac{3}{4}\beta\kappa X^2\right)\xi \\ &\quad - \frac{\beta\kappa}{8}\left[3(\xi^2 - \sigma_x)X + (\eta^2 - \sigma_y)X + \xi^3 + \xi\eta^2\right] + F_x(t) \\ &\equiv U_x(\xi, \eta, X) + F_x(t)\end{aligned} \tag{24a}$$

and

$$\begin{aligned}\dot{\eta} &= \frac{1}{2}\left(\kappa - R - \frac{\beta\kappa}{4}X^2\right)\eta - \frac{\beta\kappa}{8}\left(\frac{1}{2}\xi\eta X + \eta^3 + \xi^2\eta\right) + F_y(t) \\ &\equiv U_y(\xi, \eta, X) + F_y(t)\end{aligned} \tag{24b}$$

where the suffix i is omitted for simplicity's sake, and

$$\langle F_z(t)\rangle = 0, \qquad \langle F_{zi}(t)F_{z'j}(t')\rangle = D_z\delta_{zz'}\delta_{ij}\delta(t - t') \tag{25}$$

with $D_x = D(1 + O(N^{-1}))$ and $D_y = D$.

The evolution equation of the entire system is described in terms of a set of closed equations, (22) and (24a) and (24b). The interactions between different hierarchical subsystems are included in equations for both microscopic and macroscopic variables, as is seen in these equations.

4. CONDITIONS FOR SELF-SYNCHRONIZATION

The feedback interaction between the macroscopic and microscopic systems causes a mathematical difficulty in consistently solving Eqs. (22) and (24a) and (24b). However, when the microscopic variables rapidly vary

and follow the macroscopic variable X, the system will always be in local equilibrium. A necessary condition for the validity of the local equilibrium is that the damping constants for the evolution of the microscopic variables have the positive sign. Hence, from Eqs. (24a) and (24b),

$$\kappa - R - \tfrac{3}{4}\beta\kappa X^2 < 0 \quad \text{and} \quad \kappa - R - \tfrac{1}{4}\beta\kappa X^2 < 0$$

Thus, the condition $R > \kappa$ must hold for the local equilibrium, regardless of the value of X^2. Under the local equilibrium, the feedback loop can be decoupled and the equations become tractable. In the next order approximation the statistical distribution of the variables $\xi(t)$ and $\eta(t)$ is close to but a little different from the local distribution under $X(t)$. By adopting Kometani and Shimizu's scheme we can classify the evolution of the microscopic variables into two parts, the relaxation toward local equilibrium and the dynamical coupling with the evolution of $X(t)$.

Let the time scales for X and for ξ and η be τ_X and τ_ζ, respectively. The evolution of the microscopic variables ξ and η is given in the time interval from t to $t + s$ such as $\tau_\zeta \ll s \ll \tau_X$ by

$$\begin{aligned}\zeta(t+s; X(t+s)) &\simeq \zeta(t+s; X(t)) + s\dot{X}\,\nabla_X \zeta(t+s; X(t)) \\ &\equiv \zeta_t^\circ(s) + s\dot{X}_t\,\nabla_X \zeta_t^\circ(s) \end{aligned} \tag{26}$$

with $\zeta = \xi, \eta$. The evolution of the first term on the rhs of Eq. (26) gives the relaxation of $\zeta(t)$ toward local equilibrium and the second term concerns the dynamical coupling. The variables $\zeta_t^\circ(s)$ in Eq. (26) may be obtained as follows. In the first approximation we tentatively neglect the contribution of the dynamical coupling to the evolution of $\zeta_t^\circ(s)$ against the local relaxation. Then the evolution of $\zeta_t^\circ(s)$ is given by a linear non-Markov equation[19,21]:

$$\begin{aligned} d\zeta_t^\circ(s)/ds = \Omega_z(t)\zeta_t^\circ(s) - \int_0^s \Psi_z(s'; t)\zeta_t^\circ(s - s')\,ds' \\ + g_z(t+s) + F_z(t+s) \end{aligned} \tag{27}$$

where

$$\Omega_z \equiv \langle U_z(\xi_t^\circ, \eta_t^\circ, X(t))\zeta_t^\circ \rangle \langle \zeta_t^{\circ 2} \rangle^{-1} \tag{28a}$$

$$\Psi_z(s; t) \equiv \langle g_z(t+s)\, g_z(t) \rangle \langle \zeta_t^{\circ 2} \rangle^{-1} \tag{28b}$$

$$g_z(t+s) \equiv \exp\big[(1 - P_t)\Lambda_t s\big](1 - P_t)\Lambda_t \zeta_t^\circ \tag{28c}$$

$$P_t A \equiv \langle A\xi_t^\circ \rangle \langle \xi_t^{\circ 2} \rangle^{-1} \xi_t^\circ + \langle A\eta_t^\circ \rangle \langle \eta_t^{\circ 2} \rangle^{-1} \eta_t^\circ \tag{28d}$$

$$\begin{aligned} \Lambda_t \equiv U_x(\xi_t^\circ, \eta_t^\circ, X(t))\frac{\partial}{\partial \xi^\circ} + U_y(\xi_t^\circ, \eta_t^\circ, X(t))\frac{\partial}{\partial \eta^\circ} \\ + \frac{D}{2}\left(\frac{\partial^2}{\partial \xi^{\circ 2}} + 2\frac{\partial^2}{\partial \xi^\circ \partial \eta^\circ} + \frac{\partial^2}{\partial \eta^{\circ 2}} \right) \end{aligned} \tag{28e}$$

From Eq. (27) we obtain in the steady state

$$2\Omega_z(t)\langle \xi_t^{\circ 2}\rangle + D = 0 \tag{29}$$

We assume that N is so large that

$$\langle \xi_t^{\circ 2}\rangle \simeq \frac{1}{N}\sum_i \xi_{ti}^{\circ 2} \equiv \sigma_z^{\circ} \tag{30}$$

By assuming a Gaussian distribution for $\{\xi_i^{\circ}\}$ and $\{\eta_i^{\circ}\}$, we are able to get

$$\langle \xi_t^{\circ 4}\rangle = 3\sigma_x^{\circ 2}(t)$$

and

$$\langle \xi_t^{\circ 2}\eta_t^{\circ 2}\rangle = \sigma_x^{\circ}(t)\sigma_y^{\circ}(t)$$

Equation (28a) can be calculated as

$$\Omega_x(t) = \tfrac{1}{8}\left[4\kappa - 4R - 3\beta\kappa X^2(t) - 3\beta\kappa\sigma_x^{\circ}(t) - \beta\kappa\sigma_y^{\circ}(t)\right] \tag{31a}$$

$$\Omega_y(t) = \tfrac{1}{8}\left[4\kappa - 4R - \beta\kappa X^2(t) - \beta\kappa\sigma_x^{\circ}(t) - 3\beta\kappa\sigma_y^{\circ}(t)\right] \tag{31b}$$

Substituting Eqs. (31a) and (31b) into Eq. (29), we obtain

$$\left[4(\kappa - R) - 3\beta\kappa X^2\right]\sigma_x^{\circ} - 3\beta\kappa\sigma_x^{\circ 2} - \beta\kappa\sigma_x^{\circ}\sigma_y^{\circ} + 4D = 0 \tag{32a}$$

and

$$\left[4(\kappa - R) - \beta\kappa X^2\right]\sigma_y^{\circ} - 3\beta\kappa\sigma_y^{\circ 2} - \beta\kappa\sigma_x^{\circ}\sigma_y^{\circ} + 4D = 0 \tag{32b}$$

which immediately result in

$$\sigma_x^{\circ} = \sigma_y^{\circ} = \left\{(\kappa - R) + \left[(\kappa - R)^2 + 4\beta\kappa D\right]^{1/2}\right\}/2\beta\kappa \tag{33}$$

for $X(t) = 0$.

The synchronization is expected to occur when the steady state giving rise to $X = 0$ becomes unstable. Linear stability analysis around $X = 0$ based on Eq. (22) gives the condition for the synchronization as

$$\kappa\left[1 - \frac{\beta}{4}(3\sigma_x^{\circ} + \sigma_y^{\circ})\right] > 0 \tag{34}$$

Substituting Eq. (33) into the inequality (34) we obtain the synchronization condition for $\kappa > 0$,

$$R > \beta D \tag{35}$$

where $\beta = 4/a_0^2$.

Qualitatively speaking, the synchronization occurs only when the magnitude of the coupling among the elementary oscillators exceeds the ratio between the "energy" of thermal fluctuation D and the "energy" of the

mechanical motion, $a_0^2/4$, in the absence of the coupling. A more detailed discussion of the condition (35) will be presented in the last section.

5. THE TIME EVOLUTION OF THE ORDER PARAMETER

To obtain a clear picture of the stochastic process of the organization of a macroscopic rhythm, let us study the time dependence of the order parameter $X(t)$. For the sake of simplicity, we assume the deviations ξ and η from the order parameter $X(t)$ are so small that the linearized approximation can be utilized in Eqs. (24a) and (24b),

$$\begin{aligned} \dot{\xi} &= \frac{1}{2}\left(\kappa - R - \frac{3\beta\kappa}{4}X^2\right)\xi + F_x(t) \\ \dot{\eta} &= \frac{1}{2}\left(\kappa - R - \frac{\beta\kappa}{4}X^2\right)\eta + F_y(t) \end{aligned} \tag{36}$$

Multiplying both sides of Eqs. (36) by 2ξ or 2η, which can be obtained by integrating Eqs. (36) with regard to time, and then taking the ensemble average of the results, we obtain

$$\begin{aligned} \langle \dot{\xi}^2 \rangle &= \left(\kappa - R - \frac{3\beta\kappa}{4}X^2\right)\langle \xi^2 \rangle + D \\ \langle \dot{\eta}^2 \rangle &= \left(\kappa - R - \frac{\beta\kappa}{4}X^2\right)\langle \eta^2 \rangle + D \end{aligned} \tag{37}$$

In the presence of the local equilibrium as discussed in the preceding section the steady state solutions of Eqs. (37) become

$$\langle \xi^2 \rangle = D \Big/ \left(\frac{3\beta\kappa}{4}X^2 + R - \kappa\right)$$

and

$$\langle \eta^2 \rangle = D \Big/ \left(\frac{\beta\kappa}{4}X^2 + R - \kappa\right) \tag{38}$$

These equations denote the explicit time dependence of the internal fluctuations ξ and η upon the order parameter X. The larger the order, the smaller the internal fluctuation. In other words, as a macroscopic rhythm is organized in a system, internal fluctuations decrease.

In the following discussions we will use a simple approximation where the sample averages σ_x and σ_y are replaced by the ensemble averages $\langle \xi^2 \rangle$ and $\langle \eta^2 \rangle$, respectively. This approximation may be not too bad as long as N is large. A closed equation for the order parameter X is obtained by substituting Eqs. (38) into Eq. (22). In this treatment, the explicit time dependences of σ_x and σ_y are neglected because of local equilibrium.

Consequently, the non-Markov character to be introduced through the explicit time dependence of σ_x and σ_y in Eq. (22) disappears and one is left with a Markovian equation:

$$\dot{X} = \frac{\kappa}{2}\left\{1 - \frac{\beta D}{4}\left[3\Big/\left(\frac{3\beta\kappa}{4}X^2 - \kappa + R\right) + 1\Big/\left(\frac{\beta\kappa}{4}X^2 - \kappa + R\right)\right]\right\}X - \frac{\beta\kappa}{8}X^3 + G(t) \tag{39}$$

Within the limits of small random force $G(t)$, Eq. (39) can be evaluated by exact integration or by using the dynamical scaling theory.[20] However, the solution will be too intricate for intuitive understanding. Hereafter we will discuss the self-synchronization in simple cases where the nonlinear parameter κ is so small that the contribution of its second- or higher-order terms to the evolution of X are negligibly small. Expanding the rhs of Eq. (39) with respect to κ and retaining the result up to the first term, we obtain

$$\dot{X} = gX - hX^3 + G(t) \tag{40}$$

with $g = (\kappa/2)(1 - \beta D/R)$ and $h = \beta\kappa/8$. In the absence of the nonlinearity in the oscillators, i.e., $\kappa = 0$, the order parameter X does not grow but fluctuates around zero due to the random force $G(t)$. In case of $\kappa > 0$, there appears a systematic force, $gX - hX^3$, which gives X a time evolution to the steady state value, $\pm(g/h)^{1/2}$ or 0. Schematically this evolution can be regarded as the motion of a Brownian particle on the potential $V(X) = -\frac{1}{2}gX^2 + \frac{1}{4}hX^4$ as shown in Fig. 1. As is clearly seen, the mutual entrainment of the oscillators is achieved when $g > 0$, or when the following condition holds:

$$R > \beta D \tag{41}$$

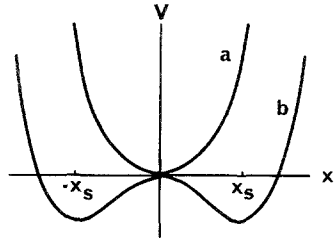

Fig. 1. A schematic representation of the macroscopic state of the system with an effective potential $V(X) = -\frac{1}{2}gX^2 + \frac{1}{4}hX^4$ for the order parameter X. The symbols a and b, respectively stand for the cases of $g < 0$ and $g > 0$. In the case of b, steady states of the system are found at $X = \pm X_s$.

which is in agreement with the condition (35). The presence of a threshold given by (41) means that the state of the elementary oscillators has two phases, the synchronized and asynchronized phases.

When $R \gg \beta D$, the coefficient g in the Langevin equation (40) becomes $\kappa/2$ and the equation is equivalent to that of a single rotating Van der Pol oscillator. The presence of βD, the effect of the external fluctuation in g, decreases the rate of the self-organization on a macroscopic rhythm by reducing the "growth rate" g by $\beta\kappa D/2R$ from that of the single oscillator. This is also the case for either Eq. (22) or Eq. (39) in which the nonlinearity is fully retained.

According to Suzuki's dynamical scaling theory, the motion of the above Brownian particle in the case $g > 0$ is denoted in terms of the time dependence of $Z = \langle X^2 \rangle$, of which the motion can be derived from Eq. (40):

$$\dot{Z} = 2gZ - 2hZ^2 + D/N \tag{42}$$

The time dependence of the variable Z can be represented in the scaling form as

$$Z = Z_s \frac{\tau}{1 + \tau} \tag{43}$$

where Z_s denotes the steady state value of Z and τ is given by the nonlinear transformation of the time:

$$Z_s = g/h + O(D/N) \tag{44}$$

and

$$\tau = (g/h)(D/2Ng + Z_0)e^{2gt} \tag{45}$$

In Eq. (45) Z_0 stands for the initial value of Z. The scaling form (43) is asymptotically correct where both the fluctuation $G(t)$ and the initial value Z_0 are small enough. Equation (45) predicts that Z will show sigmoidal growth.

To express the time needed for the temporal organization, we define an onset time t_0 which is the period from $Z = Z_0$ to $Z = Z_s/2$. The time t_0 is easily given by Eqs. (43) and (45) as

$$t_0 = -\frac{1}{2g} \ln\left[\frac{h}{g}\left(\frac{D}{2Ng} + Z_0 \right) \right] \tag{46}$$

These results will be compared with those of computer experiments in the next section.

To see the influence of the external fluctuation $G(t)$ upon the process of the self-synchronization, we derive the probability distribution function

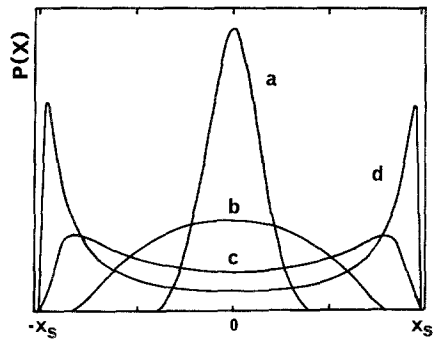

Fig. 2. A representation of the process of the self-synchronization by means of the probability distribution function $P(X;\tau')$. A disordered state a varies to ordered state d in the order of a, b, c, and d. This is essentially a replot of a figure in Ref. 20.

P_{sc} for X. By use of the scaling theory, one is able to find

$$P_{sc}(X,\tau') = \frac{1}{(2\pi\tau')^{1/2}}\left(1-\frac{h}{g}X^2\right)^{-3/2}\exp\left[-\frac{X^2}{2\tau'(1-hX^2/g)}\right] \tag{47}$$

with

$$\tau' = \left(\frac{D}{2Ng}+Z_0\right)e^{2gt} = \frac{h}{g}\tau \tag{48}$$

When the initial distribution of X has a sharp peak at $X = 0$, namely, when the microscopic oscillators are completely out of phase at $t = 0$, the initial peak gradually decreases in the following time and is decomposed into a pair of peaks symmetrically. At the final stage, the distribution of X concentrates around $X = X_s$ and $-X_s$, as shown in Fig. 2.

6. COMPUTER EXPERIMENTS

To check the validity of the above results, they will be compared with those obtained by computer experiments. Taking $\omega_0 = 1.0$, $\epsilon = 0.1$, $\beta = 1.0$, and $\kappa = 0.5$ in Eq. (1), the elementary oscillation has the amplitude $a_0 = 2.0$ and the period $T = 2\pi$ in the absence of the interaction and the fluctuation. We fix the number of the elementary oscillators to $N = 50$ and vary only R and D. In the numerical integration in the following computer experiments we used the Runge–Kutta–Gill method with a system M-200H in the Tokyo University Computer Center. The random force $f_i(t)$ is introduced in the form of the Wiener process[22] with the random number generated by the program library in the computer center.

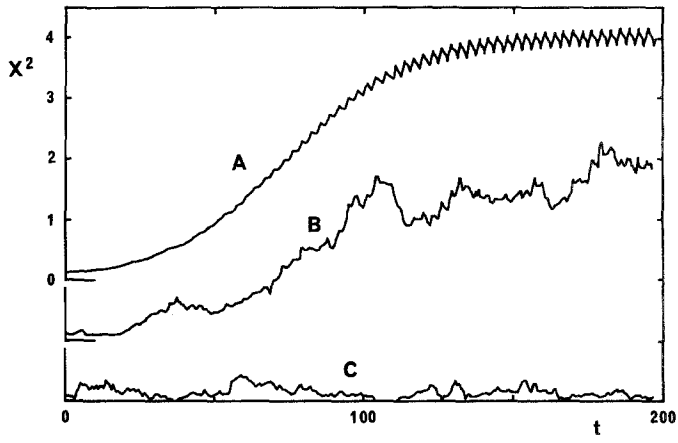

Fig. 3. Three typical cases of the self-synchronization appearing in a system of 50 Van der Pol oscillators with $\omega_0 = 1.0$, $\epsilon = 0.1$, $\beta = 1.0$, and $\kappa = 0.5$ with $R = 2.0$. The diffusion constant D due to external fluctuations is 0, 1, and 3 for plots A, B, and C, respectively. The ordinate represents the square of the order parameter, X^2, with a scale where the amplitude of the noninteracting oscillator is given by 2. The abscissa denotes the time in the scale where the period of the noninteracting oscillator is 2π. The base line for plots B and C is shifted downward by 1 and 2, respectively.

Three typical cases are demonstrated in Fig. 3 for the time course of $X^2(t)$. In the initial state, the 50 oscillators are placed on the limit cycle with a uniform phase distribution. Dissipative interactions are introduced to the elementary oscillators at the time $t = 0$. Then the time evolution of X^2 is observed using the equation $X^2 = (\sum x_i/N)^2 + (\sum y_i/N)^2$, where x_i and y_i are defined on the space fixed coordinate, for the sake of convenience. This definition of X^2 is coincident with that in Section 3 when the macroscopic oscillation has the same angular frequency ω_0 as that of the rotating coordinate. This condition has been fairly well satisfied in the region $D/R < 1$ in our computer experiments. In Fig. 3, as predicted by Eq. (43), plot A for X^2 has a sigmoidal form versus time in the absence of external fluctuation and the steady state value of X is found around 4 superimposed with a fine rapid oscillation. This rapid oscillation is caused by the fact that the Van der Pol oscillation is unharmonic and its amplitude repeatedly deviates by the factor ϵ from that of the corresponding harmonic oscillation. Weak fluctuations of $D/R = 1.0/2.0$ are introduced externally to the interacting oscillators in the case of plot B. The fluctuations markedly disturb the growth of the order parameter so that X^2 fluctuates with a long time scale of the order of more than 10^2. Plot C denotes X^2 in the case that the synchronization condition is broken ($D/R = 3.0/2.0$). In this case

X^2 remains in the neighborhood of 0 and no macroscopic oscillation appears in the system.

Figure 4 demonstrates the time dependence of the variance of the phases in plot *a* and that of the mean value of the amplitudes in plot *b*. Figures 4-1–4-3, respectively, correspond to the time courses exhibited by *A*–*C* in Fig. 3. In Fig. 4-1, the variance of the phases rapidly decreases at an early stage and simultaneously the mean value of the amplitudes also decreases. In other words, the phase coherence is attained first, and then in the later stage only the mean amplitude changes gradually and recovers the

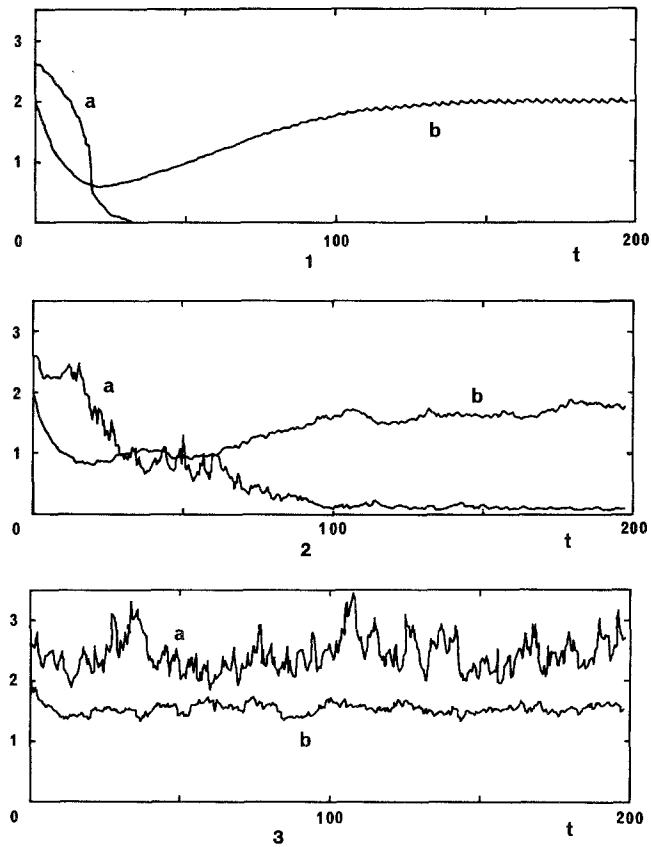

Fig. 4. The time dependence of the variance of the phases (plot *a*) and of the mean of the amplitudes (plot *b*) in the process of the self-synchronization of 50 Van der Pol oscillators. Figures 4-1, 4-2, and 4-3, respectively, correspond to the cases of *A*, *B*, and *C* in Fig. 3. The ordinate denotes the variance of the phases and the mean of the amplitudes for plots *a* and *b*, respectively. The abscissa is given as in Fig. 3.

original value of 2 for the elementary oscillation. This two-step change will be the general character of the self-synchronization caused by dissipative interactions. In the second step, the behavior of the order parameter coincides with that of the mean square of the amplitudes, since the 50 elementary oscillators are in phase in this stage. In Fig. 4-2 the variance of the phases (a) slowly decreases and remains at about 0.1 in the steady state. This deviation of phases from the complete synchronization gives the elementary oscillators a force to diminish the mean amplitude (b) through the dissipative interactions. When no macroscopic oscillation is observed as shown in Fig. 3C, the variance of the phases is found to remain at the initial level as exhibited in Fig. 4-3. The amplitudes have the mean value at 1.6 and the variance of their phases is about 0.6. In this case the phase coherence is hardly attained among the elementary oscillators. We see the state $X \simeq 0$ shown in Fig. 3C is derived from the disappearance of the phase coherency among elementary oscillators.

Figures 5 and 6 show the steady state value Z_s of $\langle X^2 \rangle$ versus the magnitude of the dissipative interaction and versus the size of the external fluctuation, respectively. Theoretical values of Z_s predicted by Eq. (44) are

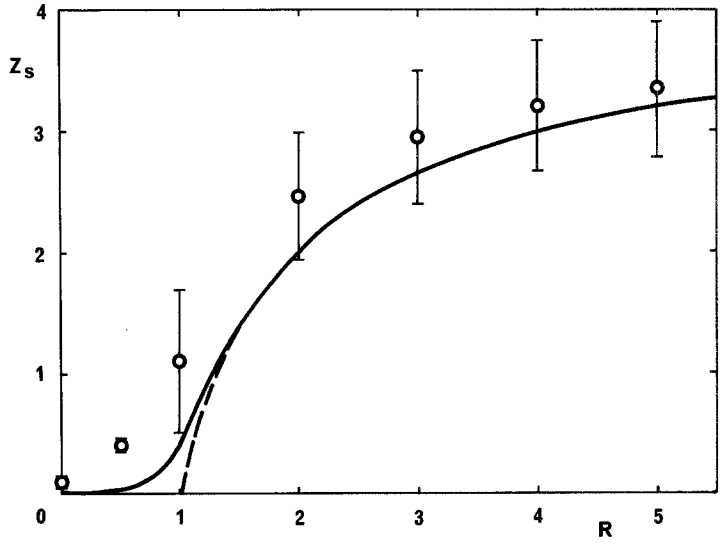

Fig. 5. The dependence of the ensemble average Z_s of the steady state values of X^2 on the magnitude of the coupling constant R of dissipative interaction, when $D = 1$ with $\beta = 1$. The solid line shows the theoretical values predicted by Eq. (44) for $N = 50$ and the broken one for $N = \infty$. The open circles indicate the values obtained in a computer experiment for $N = 50$ and the vertical lines denote the standard deviation.

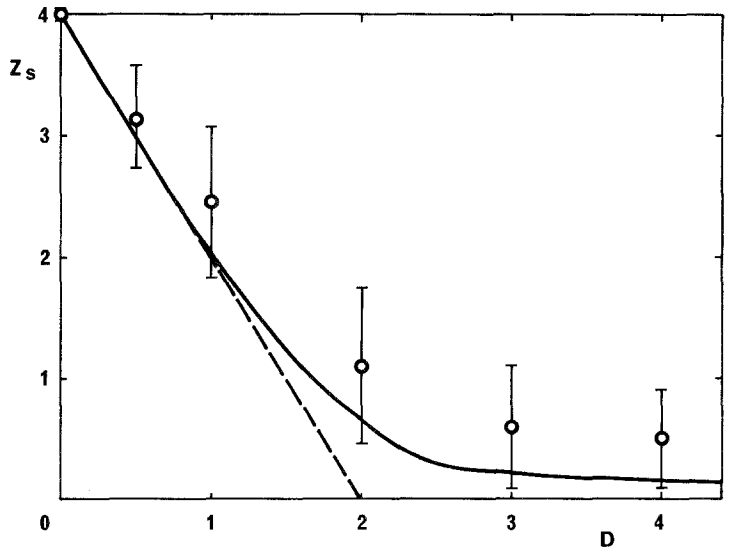

Fig. 6. The dependence of the ensemble average Z_s of the steady state values of X^2 on the magnitude of the diffusion constant D with $\beta = 1$ when the magnitude of the interaction is given as $R = 2$. The theoretical values from Eq. (44) are indicated by the solid line for $N = 50$ and by the broken line for $N = \infty$. The open circles denote the experimental values for $N = 50$ and the vertical lines show the standard deviation.

plotted by the solid line ($N = 50$) and by the broken line ($N = \infty$). The difference of Z_s between the cases of $N = 50$ and ∞ is caused by the existence of the random force $G(t)$ in Eq. (40). The open circle denotes the average of $X^2(t)$ over the time period 2500 in the steady state and the vertical line its standard deviation. These experimental results have been proved to be almost the same regardless of the initial condition, in phase or out of phase, and of the sample paths. In computer experiments, the large time scale fluctuations of X^2 are apparent as shown by the standard deviation in these figures. In the region $D/R > 1$, Z_s is found to be nearly zero. A reasonable agreement is demonstrated between the theory and computer experiments.

From the above figures we recognize two phases in the state of the correlation among the elementary oscillators. One is a disordered or temporally disorganized phase where the self-synchronization has failed. The other is an ordered or temporally organized phase where the elementary oscillators are mutually entrained. The transition between the two phases is a kind of second or continuous phase transition as shown in Figs. 5 and 6. The critical slowing down which is generally observed at the critical

condition in a phase transition has not clearly been proved because of the large time scale fluctuation in the steady state in our computer experiments.

The onset time t_0 of the self-synchronization is compared between the theory and the computer experiments in Figs. 7 and 8, where the dependences of t_0 on R and on D, respectively, are shown. Since Eq. (46) predicts t_0 with the time scale τ_1 to be a result of the multi-time-scale method, the plot is the value of t_0 given by Eq. (46) divided by $d\tau_1/dt = \epsilon$. In computer experiments, we determine t_0 for convenience as follows: first, the time when X^2 initially arrives at half of the Z_s theoretically predicted is obtained for a sample path of X^2 and then the values obtained in such a way are averaged over four examples with different paths of fluctuations and with the same initial condition mentioned in the explanation of Fig. 3. This averaged result is plotted in Figs. 7 and 8 by open circles and the vertical lines show the standard deviation among the four examples. This rough estimation of t_0 in computer simulations shows us the reasonable agreement with the theoretical value qualitatively and quantitatively.

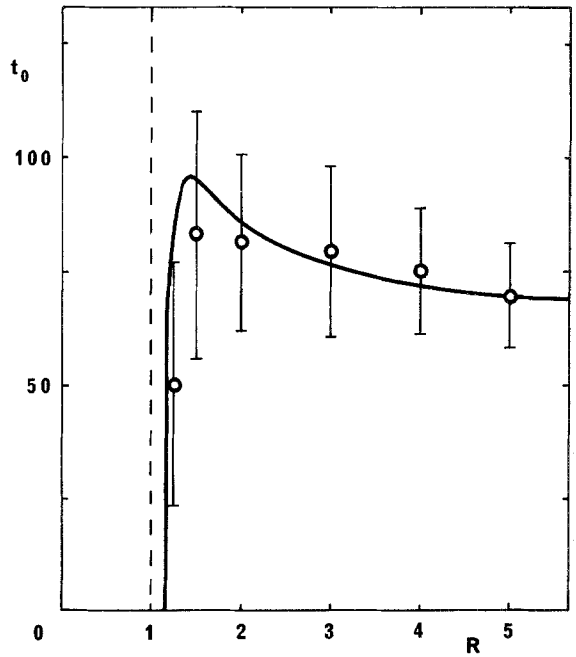

Fig. 7. The dependence of the onset time t_0 on the magnitude of coupling constant R for $D = 1$ with $\beta = 1$. The solid line indicates the theoretical values for $N = 50$ predicted from Eq. (46). The vertical broken line stands for the threshold condition for the self-synchronization. The open circles indicate t_0 obtained in a computer experiment for $N = 50$ with the standard deviation denoted by vertical lines.

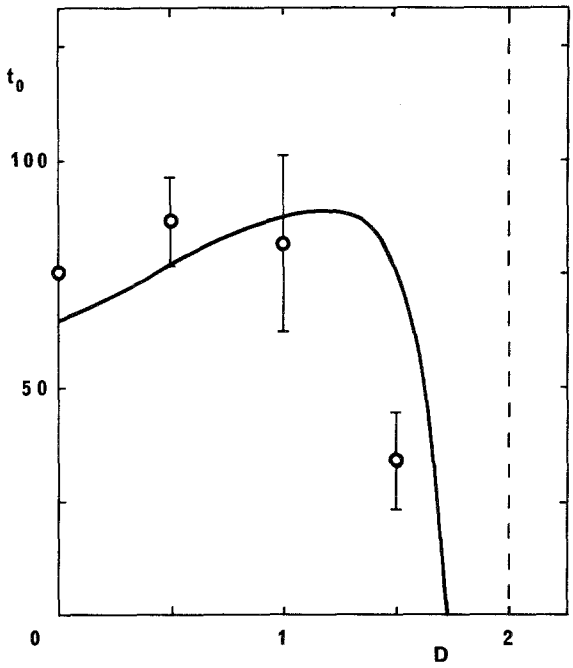

Fig. 8. The dependence of the onset time t_0 on the magnitude of the diffusion constant for the external fluctuation with $R = 2$ and $\beta = 1$. The solid line shows the theoretical values from Eq. (46) for $N = 50$. A computer experiment for $N = 50$ gave the data indicated by the open circles with the standard deviation indicated by vertical lines. The vertical broken line indicates the threshold condition for the self-synchronization.

7. DISCUSSION AND CONCLUSION

Two different approaches are presented in Sections 4 and 5 for the derivation of the synchronization condition, $R > \beta D$ for $\kappa > 0$. These treatments were based on two common assumptions: the presence of a local equilibrium and the small nonlinearity in the elementary oscillation. The former assumption required $\kappa < R$. Because of the latter assumption, a Gaussian distribution was used for ξ and η in the calculation of σ_x and σ_y in Section 4. This is asymptotically correct in the limit of small nonlinearity. In Section 5, the time evolution of the order parameter X is treated by an equation, Eq. (40), where terms higher than the first order with respect to κ are neglected. In short, our synchronization condition is valid, provided that the elementary oscillations have a small nonlinearity. This is the reason why we took $\kappa = 0.5$ in computer experiments. However, the asynchronous phase has been recognized as well as the synchronized one when the value of κ is of the order of 1. The transition between two phases is a kind of

second-order phase transition described by a Landau-type equation. A macroscopic oscillation appears as a Hopf-type bifurcation.

We have so far discussed self-synchronizations caused by dissipative interactions. In the case of the mechanical oscillation, such an interaction is caused by the friction. The electrical coupling between cardiac cells can be described by a set of equations which can be reduced to those similar to Eqs. (16).[23] That is, the coupling between excitable cells is of a dissipative type. It is of interest to qualitatively compare such synchronizations with the cases with nondissipative interactions which can be represented as $\epsilon(R/N)\sum_j(x_j - x_i)$ in the equation of motion, Eq. (1).

The multi-time-scale method applied to Eq. (1), after replacing the dissipative interaction term by nondissipative one, leads to

$$\begin{aligned} \dot{x}_i &= \frac{\kappa}{2} x_i\left[1 - \frac{\beta}{4}\left(x_i^2 + y_i^2\right)\right] - \frac{R}{2N}\sum_j (y_j - y_i) + g_{xi}(t) \\ \dot{y}_i &= \frac{\kappa}{2} y_i\left[1 - \frac{\beta}{4}\left(x_i^2 + y_i^2\right)\right] + \frac{R}{2N}\sum_j (x_j - x_i) + g_{yi}(t) \end{aligned} \tag{49}$$

The time dependences of x_i and y_i are influenced by the differences in $\{y_j\}$ and in $\{x_j\}$, respectively. Equations (49) can be further rewritten in terms of amplitude a_i and phase ϕ_i by utilizing Eqs. (A.1) in the Appendix as

$$\dot{a}_i = \frac{\kappa}{2} a_i\left(1 - \frac{\beta}{4} a_i^2\right) + \frac{R}{2} A \sin(\Phi - \phi_i) + \frac{D}{2a_i} + F_{ai}(t) \tag{50a}$$

$$\dot{\phi}_i = \frac{R}{2}\left[1 - \frac{A}{a_i}\cos(\Phi - \phi_i)\right] + \frac{F_{\phi i}(t)}{a_i} \tag{50b}$$

The interaction term $(RA/2)\sin(\Phi - \phi_i)$ in Eq. (50a) causes a decrease or increase in the amplitude a_i according to the size of $\Phi - \phi_i$, while the amplitude is never increased in the case of dissipative interactions as shown in the Appendix. Equation (50b) shows that the stable steady state is not found as $\Phi = \phi_i$, with the first approximation that $a_i = A$. Equations (50a) and (50b) show that the system with the nondissipative coupling does not give rise to so simple a synchronized state as the one in the dissipative coupling. This is consistent with the results of computer simulations by Pavlidis where quasiperiodic oscillations and randomlike variations of periods are observed.[24]

Equations of motion for coupled nonlinear oscillators can be written simply both for dissipative and nondissipative interactions as

$$\dot{W}_i = BW_i - C|W_i|^2 W_i + \Gamma\sum_j (W_j - W_i) + G_i(t) \tag{51}$$

in the complex space $W_i \equiv x_i + iy_i$. In Eq. (51) the coefficients B and C are

real quantities and the real and imaginary parts of Γ denote the dissipative and nondissipative interactions, respectively. As a causal dynamics, the stability of the synchronization was studied by Aizawa in the absence of the random fluctuating forces by using this type of equation.[25] Chemical reaction diffusion systems in the neighborhood of a soft and hard mode instability are able to be discussed with equations similar to Eq. (51).[26] In this case, the difference between the diffusion constants for two chemical substances gives the imaginary part of Γ.

Note Added in Manuscript. After writing this paper, we received a communication from Professor Y. Kuramoto of the Kyoto University and learned that he also showed the existence of the phase transition in populations of coupled limit cycle oscillators in the presence of external fluctuations. The critical condition obtained by his phenomenological analysis bears a similarity to our result.[27]

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APPENDIX

Let us derive the Langevin equations for the phase $\{\phi_i\}$ and the amplitude $\{a_i\}$ from the Fokker–Planck equation for $\{x_i\}$ and $\{y_i\}$. These variables relate to each other as follows:

$$\begin{aligned} x_i &= a_i \cos\phi_i \\ v_i/\omega_0 = y_i &= -a_i \sin\phi_i \end{aligned} \tag{A.1}$$

Since $dx_i\, dy_i = a_i\, da_i\, d\phi_i$, two distribution functions for these variables have the relation

$$P(\{a_i\}, \{\phi_i\}) = \prod_{i=1}^{N} a_i P(\{x_i\}, \{y_i\})$$

and then, corresponding Fokker–Planck operators are linked as follows:

$$\bar{H}_1(\{a_i\}, \{\phi_i\}) = \prod_{i=1}^{N} a_i \bar{H}_1 \prod_{i=1}^{N} a_i^{-1} \tag{A.2}$$

The Fokker–Planck operator for $\{a_i\}$ and $\{\phi_i\}$ can be explicitly calculated by using Eqs. (A.1), (A.2), and (15)

$$\begin{aligned}\bar{H}_1(\{a_i\},\{\phi_i\}) = \sum_i \Bigg\{ & -\frac{\kappa\omega_0}{2}\frac{\partial}{\partial a_i}a_i\left(1-\frac{\beta}{4}a_i^2\right) \\ & + \frac{\omega_0^2 D}{2}\left(\frac{\partial^2}{\partial a_i^2} - \frac{\partial}{\partial a_i}\frac{1}{a_i} + \frac{1}{a_i^2}\frac{\partial^2}{\partial \phi_i^2}\right) \\ & - \frac{\omega_0 R}{2N}\sum_j \Bigg[\frac{\partial}{\partial a_i}\left[a_j\cos(\phi_j-\phi_i) - a_i\right] \\ & + \frac{a_j}{a_i}\frac{\partial}{\partial \phi_i}\sin(\phi_j-\phi_i)\Bigg]\Bigg\} \end{aligned} \tag{A.3}$$

The Langevin equations are given by Eq. (A.3)

$$\dot{a}_i = \frac{\kappa}{2}a_i\left(1-\frac{\beta}{4}a_i^2\right) + \frac{D}{2a_i} + \frac{R}{2}\left[A\cos(\Phi-\phi_i) - a_i\right] + F_{ai}(t) \tag{A.4}$$

$$\dot{\phi}_i = \frac{A}{2a_i}\sin(\Phi-\phi_i) + \frac{1}{a_i}F_{\phi i}(t) \tag{A.5}$$

where

$$\sum x_i/N \equiv A\cos\Phi, \qquad \sum y_i/N \equiv -A\sin\Phi \tag{A.6}$$

and

$$\langle F_{bi}(t)\rangle = 0, \qquad \langle F_{bi}(t)F_{b'j}(t')\rangle = D\delta_{ij}\delta_{bb'}\delta(t-t') \tag{A.7}$$

with $b = a, \phi$. It should be noted that Eq. (A.3) is described with the time scale τ_1 whereas Eqs. (A.4), (A.5) with the time scale $\omega_0\tau_1$.

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